Lawson. Spin Geometry. Chapter 3. Index Theorem

§ 1. Differential Operators

VB E, F with $\mathbb{C}^p \searrow X \swarrow \mathbb{C}^q$. $P : \Gamma(E) \longrightarrow \Gamma(F)$ D.O. of order m

if locally $P = \sum_{|\alpha| \le m} \underbrace{A^\alpha(x)}_{q\times p \text{ matrix.}} \frac{\partial^{|\alpha|}}{\partial x^\alpha}$

Eg. $E = F = \mathbb{R}$ Laplacian $\Delta = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x_j} \left(\sqrt{g}\, g^{jk} \frac{\partial}{\partial x_k} \right)$

$= g^{jk} \frac{\partial^2}{\partial x_j \partial x_k} + \text{l.o.t.}$

Under coordi change $x_i = x_i(\tilde{x})$,

$\frac{\partial f}{\partial x} = \frac{\partial \tilde{x}}{\partial x} \frac{\partial f}{\partial \tilde{x}} \overset{\text{tensorial}}{\Longrightarrow} \mathrm{Diff}^{\le 1} \simeq \Gamma(T_X^*) + \Gamma(\mathbb{R})$ $(\otimes \mathrm{Hom}(E,F))$

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = \frac{\partial \tilde{x}_j}{\partial x_1} \frac{\partial}{\partial \tilde{x}_j} \left(\frac{\partial \tilde{x}_k}{\partial x_2} \frac{\partial f}{\partial \tilde{x}_k} \right) = \frac{\partial \tilde{x}_j}{\partial x_1} \frac{\partial \tilde{x}_k}{\partial x_2} \frac{\partial^2 f}{\partial \tilde{x}_j \partial \tilde{x}_k} + \frac{\partial \tilde{x}_j}{\partial x_1} \frac{\partial^2 \tilde{x}_k}{\partial \tilde{x}_j \partial x_2} \frac{\partial f}{\partial \tilde{x}_k}$$

$\Rightarrow$ top order part behaves as a tensor, called symbol

$(x, \xi) \in T_x^* X$, $\sigma_\xi(P) \triangleq i^m \sum_{|\alpha| = m} A^\alpha(x) \xi^\alpha : E_x \longrightarrow F_x$

(homog. of deg m in ξ)

P elliptic $\overset{\Delta}{\Longleftrightarrow} \forall \xi \in T_x^* X \setminus 0$, $\sigma_\xi(P) : E_x \xrightarrow{\cong} F_x$

Eg. $\sigma_\xi(\Delta) = -|\xi|^2 \Rightarrow$ elliptic

Eg. $D\!\!\!/ : \Gamma(S\!\!\!/) \circlearrowleft$ Dirac.

$\sigma_\xi(D\!\!\!/) = i\xi \cdot \leftarrow$ Clifford multi. $\rightsquigarrow$ elliptic.

$\rightsquigarrow i(P) \triangleq [\pi^* E, \pi^* F, \sigma(P)] \in K(T^*_{\le 1} X, T^*_{=1} X)$

§2. Sobelev theory

$P : \Gamma(E) \longrightarrow \Gamma(F)$ deg. m diff. op.

$\rightsquigarrow P : L^2_k(E) \longrightarrow L^2_{k-m}(F)$

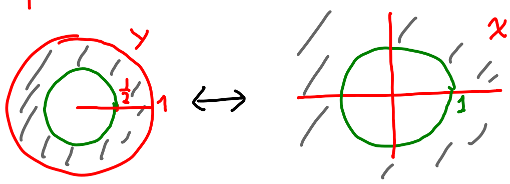
$L^2_k(E)$: completion of $\Gamma(E)$ w.r.t.

$$\|u\|^2_k = \int_X |u|^2 + \int_X |\nabla u|^2 + \dots + \int_X |\nabla^k u|^2$$

(some fix Herm. conn. on E).

Cover $X \supset \bar{U}_\beta \simeq \{|y|^2 \leq 1\}$

s.t. $B_\beta \simeq \{|y|^2 < \frac{1}{2}\}$'s still cover X ($B_\beta \subset \bar{U}_\beta$)

Change coord. 

$\Rightarrow u \in \Gamma(E) \rightsquigarrow \forall \beta$ bdd. fu. $u : \mathbb{R}^n_x \to \mathbb{C}^p$

& $|D^\alpha u(x)| (1 + |x|)^{|\alpha|} \leq C$

Fourier transf. $\hat{u}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{x \in \mathbb{R}^n} e^{-i x \cdot \xi} u(x) \, dx$

$\mathcal{S}(x) \xrightarrow[\cong]{\wedge} \mathcal{S}(\xi)$ Schwarz space

i.e. $\forall \alpha, k, |D^\alpha u(x)| \leq C_{\alpha,k} (1 + |x|)^{-k}$

(1) inversion formula. $\hat{\hat{u}} = u$

(2) exchange differentiation $\longleftrightarrow$ multi.

$$\widehat{\frac{\partial u}{\partial x_j}}(\xi) = \xi_j \hat{u}(\xi), \qquad \widehat{x_j u}(\xi) = \frac{\partial}{\partial \xi_j} \hat{u}$$

(3) isometry $\langle u, v \rangle_x = \langle \hat{u}, \hat{v} \rangle_\xi$

$\Rightarrow |u|^2_{C^k} \triangleq \sup_{\mathbb{R}^n} \sum_{|\alpha| \leq k} |D^\alpha u|^2(x) \sim \int (1 + |\xi|)^{2s} |\hat{u}| \, d\xi$

(4) exchange product $\longleftrightarrow$ convolution ($\Rightarrow L^2_s \subset\subset L^2_{s-\varepsilon}$)

- $|u(x)|^2 \leq C\left(\int |\hat{u}|\, d\xi\right)^2 \quad (\because u = \hat{\hat{u}})$

$$= C \underbrace{\int (1+|\xi|)^{-2s} d\xi}_{\leq C \text{ if } 2s > n} \times \underbrace{\int (1+|\xi|)^{2s} |\hat{u}|^2 d\xi}_{\sim \|u\|^2_{L^2_s}}$$

(Schwarz Ineq$^{\underline{t}}$) $(\Sigma a_j b_j)^2 \leq (\Sigma a_j^2)(\Sigma b_j^2)$

i.e. $|u|_{C^0} = \sup_{\mathbb{R}^n} |u| \leq C \|u\|_{L^2_s}$ if $s > \frac{n}{2}$

Similarly, $|u|_{C^k} \leq C \|u\|_{L^2_s}$ if $s \geq \frac{n}{2} + k$

$\rightsquigarrow$ Sobolev embedding theorem $L^2_s \underset{\text{cts emb.}}{\subseteq} C^k$ if $s \geq \frac{n}{2} + k$

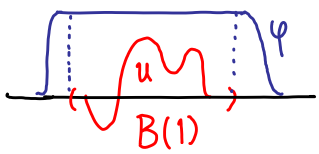
- $L^2_s \subset L^2_{s-\varepsilon}$, in fact $\subset\subset$ (i.e. 'bdd. $\longrightarrow$ cgt subseq')

Recall product $\overset{\text{Fourier}}{\longleftrightarrow}$ convolution $(\Rightarrow L^2_s \subset\subset L^2_{s-\varepsilon})$

$$\varphi * u(x) = \int \varphi(x-y)\, u(y)\, dy$$

Theorem (Rellich lemma)

$$\left.\begin{array}{l} \operatorname{Supp}(u_j) \subset B^n_{\text{fixed}} \\ \|u_j\|_{L^2_s} \leq C \end{array}\right\} \xrightarrow{\text{up to subseq.}} u_j \Longrightarrow u_\infty \text{ in } L^2_{s'} \ (\lneq s)$$

Pf: 

$u = \varphi u \Longrightarrow \hat{u} = \hat{\varphi} * \hat{u}$

$$\Rightarrow D^\alpha \hat{u}(\xi) = \int D^\alpha \hat{\varphi}(\xi - \eta)\, \hat{u}(\eta)\, d\eta$$

$$\Rightarrow \left| \text{---}\,''\,\text{---} \right|^2 \leq \underbrace{\left(\int (1+|\eta|)^{-2s} |D^\alpha \hat{\varphi}|^2 (\xi-\eta)\, d\eta\right)}_{\lesssim\ C_\alpha(\xi)} \times \underbrace{\int (1+|\eta|)^{2s} |\hat{u}|^2 d\eta}_{\|u\|^2_{L^2_s}}$$

$C_\alpha(\xi)$: unif. bdd. for ξ in cpt set

$\|u\|^2_{L^2_s} \leq C$ $\forall u_j$ (assumpt$^{\underline{n}}$)

$\Rightarrow D^\alpha \hat{u}_j$'s unif. bdd on cpt. subsets of $\mathbb{R}^n_\xi$

$\xrightarrow[\text{Ascoli}]{\text{Arzela-}}$ ($\exists$ subseq) $\hat{u}_j$'s unif. Cauchy on cpt. subsets of $\mathbb{R}^n_\xi$ $(*)$

$$\|u_j - u_k\|^2_{L^2_{\underset{\hat{s}}{s'}}} = \underbrace{\int_{|\xi|>r} \underbrace{(1+|\xi|)^{2s'}}_{\overset{\wedge\!|}{(1+r)^{-2(s-s')}(1+|\xi|)^{2s}}} |\hat{u}_j - \hat{u}_k|^2 d\xi}_{\underbrace{\le \frac{1}{r^{2(s-s')}} \underbrace{\|u_j - u_k\|^2_{L^2_s}}_{\le C \text{ (assumpt}^{n}\text{)}}}_{\le \varepsilon \text{ (by choose } r \gg 0)}} + \underbrace{\int_{|\xi|\le r} (1+|\xi|)^{2s'} |\hat{u}_j - \hat{u}_k|^2 d\xi}_{\le C \cdot \sup_{|\xi|\le r} |\hat{u}_j - \hat{u}_k| \;\nearrow\; \le \varepsilon \text{ (by } (*)).}$$

i.e. u_j's Cauchy in $L^2_{s'}$ $(<s)$ $\Rightarrow$ cgt. ($\because L^2_{s'}$ Hilb. sp.). #

Cor. $\left.\begin{array}{l} \|u_j\|_{L^2_s} \le C \\ \operatorname{Supp} u_j \subseteq B^n \end{array}\right\} \xrightarrow[\text{subseq.}]{\text{up to}} u_j \xrightarrow{C^k} u_\infty$ if $s > \frac{n}{2} + k$

Thm. $L^2_s \otimes L^2_{-s} \xrightarrow{\int uv} \mathbb{C}$ perfect pairing.

Pf: $\int u v \, dx = \int (1+|\xi|)^s \, \hat{u} \, (1+|\xi|)^{-s} \, \hat{v} \, d\xi$

$\left| \text{ — '' — } \right| \le \|u\|_{L^2_s} \cdot \|v\|_{L^2_{-s}}$

Also, "=" realized by $\hat{u} := \bar{\hat{v}} \cdot (1+|\xi|)^{-2s}$.

Prop. $P = \sum_{|\alpha|\le m} A^\alpha(x) D^\alpha$ order m diff. op. on $\mathbb{R}^n$

bdd. coeff. $\Rightarrow$ $P : L^2_s \longrightarrow L^2_{s-m}$ $\quad \forall s$

· Globalize $\mathbb{R}^n \rightsquigarrow X^n$. Use partition of unity $u = \sum_\beta \underbrace{\chi_\beta u}_{u_\beta \text{ on } \mathbb{R}^n}$

$\|u\|_{L^2_s} \triangleq \sum_\beta \|u_\beta\|_{L^2_s}$. (different choices $\Rightarrow$ equivalent $\|\cdot\|_{L^2_s}$)

§3. Pseudo-diff. operators.

$$\text{D.O. } P \xleftrightarrow{\wedge} \times\,(\text{polyn}) \quad p(x,\xi)$$

$$\psi\text{.D.O.} \xleftrightarrow{\wedge} \times\,(\text{funct}^{n})$$

$$Pu(x) = \frac{1}{(2\pi)^{n/2}} \int e^{ix\cdot\xi}\, p(x,\xi)\, \hat{u}(\xi)\, d\xi$$

Prop. $|D_x^\alpha D_\xi^\beta p(x,\xi)| \leq C_{\alpha\beta} (1+|\xi|)^{m-\beta} \quad \forall\, \alpha,\beta$

(write $p \in \mathrm{Symb}^m$)

p has cpt x-supp

$\Rightarrow P : L^2_{s+m} \longrightarrow L^2_s \quad (\in \psi DO_m)$

- If $m = \text{order } P < 0 \Rightarrow P(L_s) \subset L_{s+|m|} \rightsquigarrow$ smoothing order $|m|$
- ∞ smoothing operator $P(L_s) \subset C^\infty$
- $P \equiv P' \overset{\text{def}^n}{\Longleftrightarrow} P - P'$ ∞-smoothing

Theorem: Given $a : \mathbb{R}^{3n}_{(x,y,\xi)} \longrightarrow \mathrm{Hom}(\mathbb{C}^p, \mathbb{C}^q)$

w/ compact x-support & y-support

$$|D_x^\alpha D_y^\beta D_\xi^\gamma a| \leq C_{\alpha\beta\gamma} (1+|\xi|)^{m-|\gamma|}$$

$$Ku(x) \triangleq \frac{1}{(2\pi)^n} \iint e^{i(x-y)\cdot\xi} \cdot a \cdot u(y)\, dy\, d\xi$$

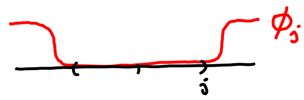
$\Rightarrow K : \mathcal{S} \circlearrowleft \psi DO$ w/ symbol $k(x,\xi)$

$$k(x,\xi) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} (D_\xi^\alpha D_y^\alpha a)(\underline{x, x}, \xi)$$

$\left(p \sim \sum_{j=1}^\infty p_j \text{ means } \forall\, m \;\exists\, l_0 \text{ s.t. } \forall\, l \geq l_0 \text{ s.t. } p - \sum^{l} p_j \in \mathrm{Symb}^{-m}\right)$

Note: $\forall$ formal $\sum_{j=1}^{\infty} p_j$ w/ $p_j \in \mathrm{Symb}^{m_j}$, $m_j \to -\infty$

$\exists \Psi DO$ w/ symbol $p \sim \sum_{j=1}^{\infty} p_j$



(reason: Makes p_j ($\rightsquigarrow \phi_j(|\xi|)\, p_j$) to 0 on $\{|\xi| \leq j\}$.

$\Rightarrow$ $p := \sum^{\infty} \phi_j\, p_j$ well-def^{d}. and ✓

Cor $a \equiv 0$ if $|x-y| \leq \varepsilon$ $\Rightarrow$ K ∞-smoothing

Cor. $\forall\, P \in \Psi DO_m$ (w/ cpt. x-support), $\forall\, \varepsilon$,

$\exists\, P' \equiv P$ s.t. $\mathrm{Supp}(Pu) \subseteq \varepsilon$-nbd. of $\mathrm{Supp}(u)$

Cor. ΨDO_m is preserved under diffeo.

($\rightsquigarrow$ globalize to mfd.).

- $P \in \Psi DO_m \Rightarrow \sigma(P) = [p] \in \mathrm{Symb}^{m} / \mathrm{Symb}^{m-1}$

  is well-def^{d} on $T^*\mathbb{R}^n$ (indep. of coord.).

  $\rightsquigarrow$ can be globalize to mfd.

§4. Elliptic operators & Parametrix.

Def. $P \in \Psi DO_m$ elliptic if

$\forall\, |\xi| \geq c > 0$ $\exists\, p(x,\xi)^{-1}$ and $|p(x,\xi)^{-1}| \leq \dfrac{C}{(1+|\xi|)^m}$.

Thm: ellptic $P \in \Psi DO_m \Rightarrow \underset{\text{(up to equiv.)}}{\exists!}\, Q \in \Psi DO_{-m}$ (parametrix)

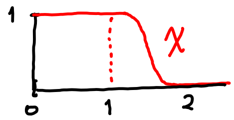
s.t. $PQ - id$ & $QP - id$: ∞-smoothing.

- In particular, $\begin{matrix} Pu \in C^\infty \\ u \in L^2_s \end{matrix} \Rightarrow u \in C^\infty$

  ($\because$ $u = Q(Pu) + Su$)

- Global ✓.

Proof of theorem:



$$q_0(x,\xi) \triangleq \chi(|\xi|)\cdot p$$

$$q_0 \in \mathrm{Symb}^{-m} \Leftarrow p \in \mathrm{Symb}^m$$

$$\rightsquigarrow Q_0 \in \Psi DO_{-m}$$

$$q_0 p - 1 = 0 = p q_0 - 1 \quad \text{outside } \{|\xi| \geq 1\}$$

$$(\not\Rightarrow Q_0 P - 1,\ P Q_0 - 1 \ \infty\text{-smoothing}$$

since $\mathrm{Symb}(QP) \neq \mathrm{Symb}\,Q \cdot \mathrm{Symb}(P)$

<u>Claim</u>: $\mathrm{Symb}(QP) = \sum_\alpha \frac{i^{|\alpha|}}{\alpha!}(D_\xi^\alpha q)\cdot(D_x^\alpha p)$

Claim $\Rightarrow$ $\mathrm{Symb}(Q_0 P - 1) \in \mathrm{Symb}^{m-1}$.

Inductively, $q \sim \sum_{\ell=0}^{\infty} q_\ell \quad (\rightsquigarrow Q \in \Psi DO)$

where $q_k \triangleq -\sum_{j=0}^{\ell-1} \sum_{|\alpha|+j=\ell} \frac{i^{|\alpha|}}{\alpha!}(D_\xi^\alpha q_j)(D_x^\alpha p)\, q_0$

Claim $\Rightarrow$ $QP - 1$ ∞-smoothing #

Pf. of claim: $P(Qu)(x) = \int e^{i x\cdot\xi} p(x,\xi) \underbrace{\widehat{Qu}(\xi)}_{?} d\xi$

$$Q = (Q^*)^* \quad \text{where} \quad \langle Qu, v\rangle_{L^2} = \langle u, Q^* v\rangle_{L^2}$$

$$q^*(y,\xi) \text{ (symbol of } Q^*) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha D_y^\alpha \overline{q(y,\xi)^t} \quad \text{(Ex)}$$

$$\Rightarrow \widehat{Qu}(\xi) = \int e^{-i y\cdot\xi} \overline{(q^*)}^t(y,\xi)\, u(y)\, dy$$

$$\Rightarrow (PQu)(x) = \iint e^{i(x-y)\cdot\xi} \underbrace{p(x,\xi)\overline{(q^*)}^t(y,\xi)}_{a(x,y,\xi)} u(y)\, dy\, d\xi$$

$$\xRightarrow{\text{earlier Thm}} \mathrm{Sym}(PQ) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha D_y^\alpha a(x,y,\xi)\big|_{x=y}$$

$$\sim \sum_\beta \frac{i^{|\beta|}}{\beta!}(D_\xi^\beta p)(D_x^\beta q) \quad \text{(by (Ex)).} \quad ▨$$

§5 Fundamental results for elliptic operators.

- S ∞-smoothing

$\Rightarrow S : L^2_s(E) \longrightarrow L^2_{s'}(E) \underset{\text{Compact}}{\subset} L^2_s(E)$ ($s' > s$)

(image of bdd seq. has cgt. subseq.)

Lemma. $H_1 \underset{Q}{\overset{T}{\rightleftarrows}} H_2$ bounded linear

s.t. $S_1 \triangleq I_{H_1} - QT$ & $S_2 \triangleq I_{H_2} - TQ$ compact

$\Rightarrow T, Q$ Fredholm (i.e. (i) $\operatorname{Ker} T$, $\operatorname{Coker} T$ finite dim, (ii) $\operatorname{Im} T \overset{\text{closed}}{\subset} H_2$)

Pf: (i) $S_1|_{\operatorname{Ker} T} = id$, S_1 cpt. $\Longrightarrow$ $\dim \operatorname{Ker} T < \infty$.

$S_2^*|_{\underbrace{\operatorname{Ker} T^*}_{\operatorname{Coker} T}} = id$, S_2 cpt $\Longrightarrow$ $\dim \operatorname{Coker} T < \infty$.

(ii) $\left.\begin{array}{l} \operatorname{Im} T \ni v_k = T u_k \\ \qquad\quad \Downarrow k\to\infty \\ H \ni v \end{array}\right\} \overset{?}{\Longrightarrow} v = Tu$?

WLOG T : 1-1

<u>Claim</u>: u_k's bounded.

$$\begin{array}{ccccc} Qv_k & = QTu_k = u_k - & \underbrace{S_1(u_k)} & & \\ \Downarrow & & \Downarrow & (\because u_k\text{'s bdd } \& \ S \text{ cpt.}) \\ Qv & & \exists\, u_\infty & (\text{up to subseq}). \end{array}$$

$TQv \Longleftarrow \underbrace{Tu_k}_{v_k \Rightarrow v} - T\underbrace{(S_1(u_k))}_{\Rightarrow u_\infty} \Rightarrow v = T(Qv + u_\infty)$.

QED.

Pf. of claim: If not, (subseq) $\|u_k\| \to \infty$,

$$\lim_{k\to\infty} T\left(\frac{u_k}{\|u_k\|}\right) = \lim_{k\to\infty} \frac{Tu_k = v_k \Rightarrow v}{\|u_k\| \to \infty} = 0$$

$$\xRightarrow{QT = I - S_1} \lim_{k\to\infty}\left(\frac{u_k}{\|u_k\|}\right) = \lim_{k\to\infty} S_1\left(\frac{u_k}{\|u_k\|}\right) = w \ \exists \quad \left(\because S_1 \text{ cpt. } u_k/\|u_k\|\text{'s bdd.}\right)$$

$\|w\| = 1$

$\xRightarrow{\text{cts.}} Tw = 0 \xRightarrow{\operatorname{Ker} T = 0} (\divideontimes)$.

Theorem X cpt. mfd. $P : \Gamma(X,E) \to \Gamma(X,F)$

P elliptic operator, order $m > 0$

(i) $\left.\begin{array}{l} Pu \in C^\infty \\ u \in L_2^s \end{array}\right\} \Rightarrow u \in C^\infty$

(ii) $P : L_s^2 \longrightarrow L_{s-m}^2$ Fredholm.

(iii) $\|u\|_{L_m^2} \leqslant C_s \left(\|u\|_{L^2} + \|Pu\|_{L^2} \right)$

$\text{Index } P := \dim \operatorname{Ker} P - \dim \operatorname{Ker} P^*$ (indep. of s for L_s^2.)

Theorem. If $P = P^*$, then (iv) $\Gamma(X,E) = \operatorname{Ker} P \overset{\perp}{\oplus} \operatorname{Im} P$.

(Cor. Hodge Decomp. $\Omega^p(X) = \underbrace{H^p(X)}_{\operatorname{Ker}\Delta} \overset{\perp}{\oplus} \operatorname{Im} d \overset{\perp}{\oplus} \operatorname{Im} d^*$)

(v) $\exists$ deg $-m$, $G : \Gamma(X,E) \circlearrowleft$ Green operator

s.t. $PG = GP = id - H$ orthogonal proj. to $\operatorname{Ker} P$.

(vi) Eigenspace $E_\lambda \triangleq \operatorname{Ker}(P - \lambda I)$ has $\dim E_\lambda < \infty$

$\subseteq C^\infty$

λ's real, discrete;

"λ's $\to \infty$", i.e. $\dim \bigoplus_{|\lambda| \leqslant \Lambda} E_\lambda \leqslant C \Lambda^{n(n+2m+2)/2m}$

$L^2(X,E) = \bigoplus_\lambda E_\lambda$ i.e. complete o.n. system

§ 6 Heat Kernel & index.

$P : \Gamma(X, E) \circlearrowleft$ pos., $P = P^*$, order $m > 0$
$\uparrow$ compact.

$\rightsquigarrow$ $L^2(X,E)$ has complete o.n. basis $\{u_k\}_{k=1}^{\infty}$

$$P u_k = \lambda_k u_k \ , \quad 0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$$

$$K_t(x,y) \triangleq \sum_{k=1}^{\infty} e^{-\lambda_k t} u_k(x) \otimes u_k^*(y) \in \Gamma(X \times X, E \boxtimes E^*)$$

Heat Kernel.

<u>Claim</u>: $\forall$ cgt. unif. in C^r on $\underset{\text{cpt.}}{I} \times X \times X$, $I \subset (0,\infty)$.

Claim $\Rightarrow$ $e^{-tP} : \Gamma(X,E) \circlearrowleft$ ∞-smoothing

where $(e^{-tP} u)(x) \triangleq \int_X K_t(x,y) u(y) dy$

- $\forall$ $u \in L_s^2(E)$, define $u(t,x) \triangleq (e^{-tP}u)(x) \in C^\infty$

s.t. $\quad \dfrac{\partial u}{\partial t} + Pu = 0 \quad \& \quad u|_{t=0} = u$

$$\mathrm{Tr}(e^{-tP}) = \int_X \mathrm{Tr}_{E_x} K_t(x,x)\, dx = \sum_{k=1}^{\infty} e^{-\lambda_k t}.$$

well-def^{d} & analy $\forall$ $t > 0$.

Remark: For $P : \Gamma(E) \longrightarrow \Gamma(F)$ elliptic,

$$\mathrm{Index}\, P = \mathrm{Tr}\, e^{-tP^*P} - \mathrm{Tr}\, e^{-tPP^*}$$

since P^*P and PP^* have exactly the same <u>nonzero</u> eigenvalues.

- $\deg P = 1 \Rightarrow \mathrm{Tr}\, K_t(x,x) \overset{t \downarrow 0}{\sim} \sum_{k=0}^{\infty} \underbrace{\rho_k(x)}_{\text{local}} t^{(k-n)/2}$

local, explicitly computable in terms of geometry of X & P.